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A Globally Convergent Method for Semi-Infinite Linear Programming

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Abstract. This paper presents a globally convergent method for solving a general semi-infinite linear programming problem. Some important features of this method include: It can solve a semi-infinite linear program having an unbounded feasible region. It requires an inexact solution to a nonlinear subproblem at each iteration. It allows unbounded index sets and nondifferentiable constraints. The amount of work at each iteration k does not increase with k.

Key words: Semi-infinite linear programming, optimality, convergence.

1. Introduction

The primal problem of semi-infinite linear programming is defined as

$$\begin{array}{lll} \text{(SIP)} & \text{maximize} & c^T x \\ & \text{subject to} & a(u)^T x - b(u) \leq 0 \quad \text{for all } u \in U, \end{array}$$

where c, $x \in R^n$, $U \subseteq R^m$ is an index set containing infinitely many points, $a : U \to R^n$, and $b : U \to R^1$. Without loss of generality, we assume that c is a unit vector, $a(u) \neq 0$ for all $u \in U$, and $\sup\{\|(a(u), b(u))\|_\infty : u \in U\} < \infty$.

There are many practical as well as theoretical problems in which the constraints depend on time or space and thus can be formulated as semi-infinite programs. The question of how to compute numerically a solution of a semi-infinite program has received increasing attention (see, e.g., Ferris and Philpott (1989), Hu (1990), Kortanek and No (1993), and Todd (1994)). For a recent extensive survey on semi-infinite programming theory, methods, and applications, one may consult Hettich and Kortanek (1993). Some common restrictions on (SIP) imposed by most existing methods are that the feasible region must be bounded and the index set U must be compact and have a nice structure. Many methods also require, at each iteration k, finding an exact solution of the nonlinear program $\sup\{a(u)^T x^k - b(u) : u \in U\}$. In this paper we present a globally convergent method for solving a general semi-infinite linear programming problem. Some important features of this method include: It can solve a semi-infinite linear program having an unbounded feasible region. It requires an inexact solution to a nonlinear subproblem at each iteration. It allows unbounded index sets (e.g., $U = \{1, 2, \dots\}$) and nondifferentiable constraints. The amount of work at each iteration k does not increase with k.

2. Preliminaries

Let $S = \{x : a(u)^T x - b(u) \leq 0 \text{ for all } u \in U\}$ denote the feasible region of (SIP), $v^* = \sup\{c^T x : x \in S\}$ denote the optimal value of (SIP), and $S^* = \{x \in S : c^T x = v^*\}$ denote the set of optimal solutions of (SIP). We assume that S is nonempty, which can be verified by solving a phase (I) problem (Gustafson 1983). Note that S^* may be empty if S is unbounded.

Let $\|\cdot\|$ be the Euclidean norm and $d(x) = \min\{\|x - y\| : y \in S\}$ be the Euclidean distance from x to S.

For any real number t, we define

$$t^+ = \begin{cases} t, & \text{if } t \geq 0; \\ 0, & \text{if } t < 0. \end{cases}$$

Let $r(x) = \sup\{(a(u)^T x - b(u))^+ : u \in U\}$ be the "biggest violation" by x. It is easy to verify that (i) $0 \leq r(x) < \infty$ for all $x \in R^n$, (ii) $r(x) = 0$ if and only if $x \in S$, (iii) $r(x) \geqslant \sup\{a(u)^T x - b(u) : u \in U\}$ for all $x \in R^n$, (iv) $r(x) = \sup\{a(u)^T x - b(u) : u \in U\}$ if $x \notin S$, and (v) $r(x)$ is a continuous convex function on R^n.

$r(x)$ measures how much x violates the constraints and $d(x)$ measures how far x is from the feasible region. If the feasible region S is ill-conditioned, then it is possible to find a sequence $\{x^k : k = 1, 2, \ldots\}$ such that $\lim_{k\to\infty} r(x^k) = 0$ and $\lim_{k\to\infty} d(x^k) = \infty$ (Hu and Wang 1989). In this situation, the task of computing numerically a solution to (SIP) becomes very difficult. Hence, in the rest of our discussion, we assume that S satisfies the following condition:

CONDITION A. There exists a constant $\tau > 1$ such that $d(x) \leq \tau r(x)$ for all $x \in R^n$.

The existence and computation of τ are discussed in Hu and Wang (1989). For example, if there exists a unit vector $\bar{x}$ and a positive number β such that $a(u)^T \bar{x} \geq \beta$ for all $u \in U$, then $\tau = \beta^{-1}$. Note that S is unbounded in this case. For a second example, if S is bounded by M and $b(u) \geq \delta > 0$ for all $u \in U$, then $\tau = \delta^{-1} M$. The existence of a relatively small τ ensures that if x almost satisfies the constraints, then x is close to (in Euclidean distance) the feasible region.

The basic idea of our method is: Let $\epsilon_k > 0$, $\lambda_k > 0$, and x^k be the current iterate. Find a constraint whose index u^k is an ϵ_k solution of the nonlinear program $\sup\{a(u)^T(x^k + \lambda_k c) - b(u) : u \in U\}$. If $x^k + \lambda_k c$ satisfies this constraint, then $x^k + \lambda_k c$ is close enough to the feasible region and the method can focus on improving the objective function value by letting $x^{k+1} = x^k + \lambda_k c$. Otherwise, the method takes care of optimality and feasibility by letting x^{k+1} be the projection of $x^k + \lambda_k c$ on $a(u^k)^T x = b(u^k)$. The sequence $\{x^k : k = 1, 2, \ldots\}$ will converge to an optimal solution of (SIP) if an optimal solution exists and the sequences

$\{\lambda_k > 0 : k = 1, 2, \dots\}$ and $\{\epsilon_k > 0 : k = 1, 2, \dots\}$ satisfy the following condition.

CONDITION B.
(B1) $\sum_{k=1}^{\infty} \lambda_k = \infty$;
(B2) $\sum_{k=1}^{\infty} \lambda_k^2 < \infty$;
(B3) $\lambda_i/\lambda_j \le j - i + 1$ for all $j \ge i \ge 1$;
(B4) $\lim_{k\to\infty} \epsilon_k/\lambda_k = 0$.

For example, if $\lambda_k = 1/k$ and $\epsilon_k = 1/k^2$ for all $k = 1, 2, \dots$, then Condition B is satisfied. (B1) and (B2) are typical assumptions of the subgradient method (see, e.g., Bazaraa *et al.* (1993)). (B2) implies that $\lim_{k\to\infty} \lambda_k = 0$ and $\sup\{\lambda_k : k = 1, 2, \dots\} < \infty$. (B3) plays an important role in the convergence proof of our method. (B2) and (B4) imply that $\lim_{k\to\infty} \epsilon_k = 0$. If Condition A and Condition B are satisfied, then the entire sequence $\{x^k : k = 1, 2, \dots\}$ generated by the method will converge to an optimal solution of (SIP) as long as an optimal solution exists. This method is inspired by the subgradient method.

3. The Method

Let $\{\lambda_k > 0 : k = 1, 2, \dots\}$ and $\{\epsilon_k > 0 : k = 1, 2, \dots\}$ satisfy Condition B.

THE METHOD.
Step 0.
Let $k := 1$;
let x^1 be an arbitrary point in R^n;
let α be a constant satisfying $0 < \alpha < 1$.
Step 1.
Find a $\bar{u} \in U$ such that
$a(\bar{u})^T(x^k + \lambda_k c) - b(\bar{u}) \ge \sup\{a(u)^T(x^k + \lambda_k c) - b(u) : u \in U\} - \epsilon_k$;
if $a(\bar{u})^T(x^k + \lambda_k c) - b(\bar{u}) \le 0$ or $\epsilon_k \le \alpha(a(\bar{u})^T(x^k + \lambda_k c) - b(\bar{u}))$,
then let $u^k := \bar{u}$, go to Step 3;
otherwise, go to Step 2.
Step 2.
Let $\epsilon_k := \alpha(a(\bar{u})^T(x^k + \lambda_k c) - b(\bar{u}))$;
find a $u^k \in U$ such that
$a(u^k)^T(x^k + \lambda_k c) - b(u^k) \ge \sup\{a(u)^T(x^k + \lambda_k c) - b(u) : u \in U\} - \epsilon_k$,
go to Step 3.
Step 3.
If $a(u^k)^T(x^k + \lambda_k c) - b(u^k) \le 0$, then let $x^{k+1} := x^k + \lambda_k c$;
otherwise, let
$x^{k+1} := x^k + \lambda_k c - (a(u^k)^T(x^k + \lambda_k c) - b(u^k))a(u^k)/\|a(u^k)\|^2$;
$k := k + 1$, go to Step 1. □

To implement the method, one needs to find an ϵ_k-optimal solution of the subproblem $\sup\{a(u)^T(x^k + \lambda_k c) - b(u) : u \in U\}$ at each iteration. Thus, the practical applicability of the method is restricted to problems where the subproblems can be solved efficiently. The constant parameter α can be specified according to the efficiency of the procedure used for solving the subproblem, i.e., a smaller α for a more efficient procedure. In Step 3, if $x^{k+1} = x^k + \lambda_k c$, we say that x^{k+1} is type 1; otherwise, x^{k+1} is the projection of $x^k + \lambda_k c$ on $a(u^k)^T x = b(u^k)$ and we say that x^{k+1} is type 2. The following facts will be used in the convergence proof. Note that Fact 2 is guaranteed by Step 2 of the method.

FACT 1. If x^{k+1} is type 2, then for all $y \in S$,

$$||x^{k+1} - y||^2 \leq ||x^k + \lambda_k c - y||^2 - ||x^{k+1} - (x^k + \lambda_k c)||^2.$$

FACT 2. If x^{k+1} is type 2, then $\epsilon_k \leq \alpha r(x^k + \lambda_k c)$.

First, let's discuss a special case. If $x^{k+1} = x^k$ for a certain k, then x^{k+1} is type 2, $a(u^k)$ is parallel to c, and thus $c^T x^{k+1} \geq v^*$ and $d(x^{k+1}) \leq (1-\theta)^{-1}\lambda_k$ (where $0 < \theta < 1$, see Lemma 1). Hence, x^{k+1} is an approximate solution of (SIP) if $x^{k+1} = x^k$ and λ_k is sufficiently small, and x^{k+1} is an exact optimal solution of (SIP) if $x^{k+1} = x^k = \bar{x}$ for infinitely many k. Next, we consider the general case and we show that the sequence $\{x^k : k = 1, 2, \dots\}$ generated by the method converges to an optimal solution of (SIP) if $S^* \neq \emptyset$.

LEMMA 1. *Let $\{x^k : k = 1, 2, \dots\}$ be generated by the method. There exists a constant θ, $0 < \theta < 1$, and a positive integer K such that $d(x^{k+1}) \leq \theta d(x^k) + \lambda_k$ for all $k \geq K$.*

Proof. For all $k = 1, 2, \dots$, let y^k be the point in S nearest to x^k and z^k be the point in S nearest to $x^k + \lambda_k c$, i.e., $d(x^k) = ||x^k - y^k||$ and $d(x^k + \lambda_k c) = ||x^k + \lambda_k c - z^k||$. Let $M = \max\{1, \sup\{\, ||a(u)|| : u \in U\}\}$ and K be sufficiently large such that $\tau\epsilon_k \leq \lambda_k$ for all $k \geq K$. The existence of such a K is guaranteed by Condition A and Condition (B4). If $k \geq K$ and x^{k+1} is type 2, then

$$\begin{aligned}
||x^{k+1} - (x^k + \lambda_k c)|| &= \frac{a(u^k)^T(x^k + \lambda_k c) - b(u^k)}{||a(u^k)||} \\
&\geq \frac{\sup\{a(u)^T(x^k + \lambda_k c) - b(u) : u \in U\} - \epsilon_k}{||a(u^k)||} \\
&= \frac{r(x^k + \lambda_k c) - \epsilon_k}{||a(u^k)||} \\
&\geq \frac{(1-\alpha) r(x^k + \lambda_k c)}{M} \qquad \text{(Fact2)} \qquad (1) \\
&\geq (1-\alpha)\tau^{-1} M^{-1} d(x^k + \lambda_k c) \qquad \text{(ConditionA)}.
\end{aligned}$$

It follows from (1) and Fact 1 that

$$\begin{aligned} d(x^{k+1})^2 &\le \|x^{k+1} - z^k\|^2 \\ &\le \|x^k + \lambda_k c - z^k\|^2 - \|x^{k+1} - (x^k + \lambda_k c)\|^2 \\ &\le d(x^k + \lambda_k c)^2 - (1-\alpha)^2 \tau^{-2} M^{-2} d(x^k + \lambda_k c)^2 \\ &= (1 - (1-\alpha)^2 \tau^{-2} M^{-2}) d(x^k + \lambda_k c)^2. \end{aligned}$$

Let $\theta = (1 - (1-\alpha)^2\tau^{-2}M^{-2})^{\frac{1}{2}}$. It is obvious that $0 < \theta < 1$. Since $d(x^k + \lambda_k c) \le \|x^k + \lambda_k c - y^k\| \le d(x^k) + \lambda_k$, we have proved $d(x^{k+1}) \le \theta d(x^k) + \lambda_k$ provided that x^{k+1} is type 2 and $k \ge K$. If x^{k+1} is type 1 and $d(x^{k+1}) = 0$, then $0 = d(x^{k+1}) \le \theta d(x^k) + \lambda_k$. If x^{k+1} is type 1, $d(x^{k+1}) > 0$, and $k \ge K$, then

$$\begin{aligned} d(x^{k+1}) &= d(x^k + \lambda_k c) \\ &\le \tau r(x^k + \lambda_k c) \qquad \text{(ConditionA)} \\ &= \tau \sup\{a(u)^T(x^k + \lambda_k c) - b(u) : u \in U\} \\ &\le \tau \epsilon_k \\ &\le \theta d(x^k) + \lambda_k. \end{aligned}$$

□

LEMMA 2. *Let $\{x^k : k = 1, 2, \dots\}$ be generated by the method. Let θ and K be defined as in Lemma 1.*
(a) For all $k \ge K$ and $m \ge 0$, $d(x^{k+m}) \le \theta^m d(x^k) + (1-\theta)^{-2}\lambda_{k+m-1}$.
(b) $\lim_{k\to\infty} d(x^k) = 0$.

Proof. As (a) is obvious in the case $m = 0$, we now assume $m \ge 1$ and $k \ge K$. Applying Lemma 1 repeatedly, we have

$$d(x^{k+m}) \le \theta^m d(x^k) + \sum_{n=1}^{m} \theta^{n-1}\lambda_{k+m-n}.$$

And,

$$\begin{aligned} &\sum_{n=1}^{m} \theta^{n-1}\lambda_{k+m-n} \\ &= \lambda_{k+m-1} \sum_{n=1}^{m} \theta^{n-1}\lambda_{k+m-n}/\lambda_{k+m-1} \\ &\le \lambda_{k+m-1} \sum_{n=1}^{m} \theta^{n-1}(k+m-1 - (k+m-n) + 1) \qquad \text{(Condition (B3))} \\ &\le \lambda_{k+m-1} \sum_{n=1}^{\infty} n\theta^{n-1} \\ &= (1-\theta)^{-2}\lambda_{k+m-1}. \end{aligned}$$

Consequently,

$$d(x^{k+m}) \le \theta^m d(x^k) + (1-\theta)^{-2}\lambda_{k+m-1} \quad \text{for all } m = 0, 1, 2, \dots .$$

It is easy to see that (b) follows from (a). □

LEMMA 3. *Let* $\{x^k : k = 1, 2, \dots\}$ *be generated by the method. If the feasible region is bounded, then there exists a subsequence* $\{x^{k_j} : j = 1, 2, \dots\}$ *that converges to an optimal solution of (SIP).*

Proof. Since S is bounded and $d(x^k) \to 0$, the sequence $\{x^k : k = 1, 2, \dots\}$ is bounded and any accumulation point of the sequence is feasible. Hence, $\limsup_{k\to\infty} c^T x^k \leq v^*$, where v^* denotes the optimal value of (SIP). Since S is bounded, v^* is finite and can be attained, i.e., there exists an $x^* \in S$ such that $c^T x^* = v^*$. If $\limsup_{k\to\infty} c^T x^k < c^T x^*$, then there exists an integer n_1 and a small positive number δ such that $c^T x^k \leq c^T x^* - \delta$ for all $k \geq n_1$. If x^{k+1} is type 2 and $k \geq n_1$, then

$$\begin{aligned}
&\|x^{k+1} - x^*\|^2 \\
&\leq \|x^k + \lambda_k c - x^*\|^2 - \|x^{k+1} - (x^k + \lambda_k c)\|^2 \\
&= \|x^k - x^*\|^2 - \|x^{k+1} - x^k\|^2 + 2\lambda_k c^T (x^{k+1} - x^*) \\
&\leq \|x^k - x^*\|^2 - 2\lambda_k \delta.
\end{aligned}$$

If x^{k+1} is type 1 and $k \geq n_1$, then

$$\begin{aligned}
\|x^{k+1} - x^*\|^2 &= \|x^k + \lambda_k c - x^*\|^2 \\
&= \|x^k - x^*\|^2 + \lambda_k^2 + 2\lambda_k c^T (x^k - x^*) \\
&\leq \|x^k - x^*\|^2 + \lambda_k^2 - 2\lambda_k \delta.
\end{aligned}$$

Therefore, for all $k \geq n_1$

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 + \lambda_k^2 - 2\lambda_k \delta. \tag{2}$$

Adding (2) from $k = n_1$ to any $n > n_1$, we have

$$\|x^{n+1} - x^*\|^2 \leq \|x^{n_1} - x^*\|^2 + \sum_{k=n_1}^{n} \lambda_k^2 - 2\delta \sum_{k=n_1}^{n} \lambda_k.$$

As $\sum_{k=1}^{\infty} \lambda_k^2$ is convergent and $\sum_{k=1}^{\infty} \lambda_k$ is divergent, the above inequality cannot hold for a sufficiently large n. Consequently, we must have $\limsup_{k\to\infty} c^T x^k = c^T x^*$. The lemma then follows easily from the boundedness of $\{x^k : k = 1, 2, \dots\}$. □

THEOREM 1. *If the feasible region is bounded, then the sequence* $\{x^k : k = 1, 2, \dots\}$ *generated by the method converges to an optimal solution of (SIP).*

Proof. Let θ, K, and y^k be defined as in Lemma 1 and $\lambda_0 = \sup\{\lambda_k : k = 1, 2, \dots\}$. We know, from Lemma 3, that there exists a subsequence $\{x^{k_j} : j = 1, 2, \dots\}$ such that $\lim_{j\to\infty} x^{k_j} = x^*$ and x^* is an optimal solution of (SIP). It remains to show that the entire sequence converges to x^*. By Lemma 2 (b), Lemma 3, and Condition (B2), for any given $\delta > 0$, there exists an integer $J > 0$ such that for all $j \geq J$ we have

$$d(x^{k_j}) < \delta^2 \lambda_0^{-1}(1-\theta)/8, \tag{3}$$

$$k_j \geq K \text{ and } \|x^{k_j} - x^*\|^2 < \delta^2/4, \text{ and} \tag{4}$$

$$\sum_{i=k_j-1}^{\infty} \lambda_i^2 < \delta^2(1-\theta)^2/8. \tag{5}$$

Now let $N = k_J$ and $n \geq N$. If $n = k_j$ for some $j \geq J$, then $\|x^{k_j} - x^*\|^2 < \delta^2/4 < \delta^2$. Otherwise, $k_j < n < k_{j+1}$ for some $j \geq J$ and let $l = n - k_j > 0$. If x^{k+1} is type 2 and $k \geq K$, then

$$\begin{aligned}
\|x^{k+1} - x^*\|^2 &\leq \|x^k + \lambda_k c - x^*\|^2 - \|x^{k+1} - (x^k + \lambda_k c)\|^2 \\
&= \|x^k - x^*\|^2 - \|x^{k+1} - x^k\|^2 + 2\lambda_k c^T(x^{k+1} - x^*) \\
&\leq \|x^k - x^*\|^2 + 2\lambda_k c^T(x^{k+1} - y^{k+1}) + 2\lambda_k c^T(y^{k+1} - x^*) \\
&\leq \|x^k - x^*\|^2 + 2\lambda_k \|c\|\, \|x^{k+1} - y^{k+1}\| \\
&= \|x^k - x^*\|^2 + 2\lambda_k d(x^{k+1}) \\
&\leq \|x^k - x^*\|^2 + 2\lambda_k(\theta d(x^k) + \lambda_k) \\
&\leq \|x^k - x^*\|^2 + 2\lambda_k d(x^k) + 2\lambda_k^2.
\end{aligned}$$

If x^{k+1} is type 1, then

$$\begin{aligned}
\|x^{k+1} - x^*\|^2 &= \|x^k + \lambda_k c - x^*\|^2 \\
&= \|x^k - x^*\|^2 + \lambda_k^2 + 2\lambda_k c^T(x^k - x^*) \\
&= \|x^k - x^*\|^2 + \lambda_k^2 + 2\lambda_k c^T(x^k - y^k) + 2\lambda_k c^T(y^k - x^*) \\
&\leq \|x^k - x^*\|^2 + \lambda_k^2 + 2\lambda_k d(x^k).
\end{aligned}$$

Hence, for all $k \geq K$

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 + 2\lambda_k^2 + 2\lambda_k d(x^k). \tag{6}$$

Adding (6) from $k = k_j$ to $k = n - 1 = k_j + l - 1$,

$$\|x^n - x^*\|^2 \leq \|x^{k_j} - x^*\|^2 + 2\sum_{m=0}^{l-1} \lambda_{k_j+m}^2 + 2\sum_{m=0}^{l-1} \lambda_{k_j+m} d(x^{k_j+m}). \tag{7}$$

From (3), (5), and Lemma 2 (a),

$$\sum_{m=0}^{l-1} \lambda_{k_j+m} d(x^{k_j+m})$$

$$\begin{aligned}
&\leq \sum_{m=0}^{l-1} \lambda_{k_j+m}(\theta^m d(x^{k_j}) + (1-\theta)^{-2}\lambda_{k_j+m-1}) \\
&\leq d(x^{k_j})\lambda_0 \sum_{m=0}^{l-1} \theta^m + (1-\theta)^{-2} \sum_{m=0}^{l-1} \lambda_{k_j+m}\lambda_{k_j+m-1} \\
&\leq d(x^{k_j})\lambda_0(1-\theta)^{-1} + (1-\theta)^{-2} \sum_{m=0}^{\infty} (\lambda_{k_j+m}^2 + \lambda_{k_j+m-1}^2)/2 \\
&< \delta^2/4. \qquad (8)
\end{aligned}$$

Finally, it follows from (4), (5), (7), and (8) that $\|x^n - x^*\| < \delta$ for all $n \geq N$. □

We have proved the convergence of the method for a bounded feasible region. The boundedness of S ensures that v^* is finite and can be attained. However, if S is unbounded, then it is possible that (i) v^* is infinite, (ii) v^* is finite and can be attained, and (iii) v^* is finite but cannot be attained. The next theorem tells us that in all three cases, the sequence $\{x^k : k = 1, 2, \dots\}$ generated by the method has the property that $\limsup_{k\to\infty} c^T x^k = v^*$ and $\lim_{k\to\infty} d(x^k) = 0$ (Lemma 2 (b)).

THEOREM 2. *Let* $\{x^k : k = 1, 2, \dots\}$ *be generated by the method. If the feasible region is unbounded, then* $\limsup_{k\to\infty} c^T x^k = v^*$ *and* $\lim_{k\to\infty} d(x^k) = 0$.

Proof. Let y^k be the point in S nearest to x^k. By Lemma 2 (b) and the feasibility of y^k,

$$\begin{aligned}
\limsup_{k\to\infty} c^T x^k &= \limsup_{k\to\infty} (c^T x^k - c^T y^k + c^T y^k) \\
&\leq \limsup_{k\to\infty} (c^T x^k - c^T y^k + v^*) = v^*.
\end{aligned}$$

If $\limsup_{k\to\infty} c^T x^k < v^*$, then there exists a point $\bar{x}$ in S and a positive number δ such that

$$\limsup_{k\to\infty} c^T x^k + \delta \leq c^T \bar{x}. \qquad (9)$$

By the definition of $\limsup_{k\to\infty} c^T x^k$, there exists an integer n_1 such that for all $k \geq n_1$

$$c^T x^k \leq \limsup_{k\to\infty} c^T x^k + \delta/2. \qquad (10)$$

It follows from (9) and (10) that for all $k \geq n_1$

$$c^T x^k - c^T \bar{x} \leq \limsup_{k\to\infty} c^T x^k + \delta/2 - (\limsup_{k\to\infty} c^T x^k + \delta) = -\delta/2.$$

As in the proof of Lemma 3 (replace x^* by $\bar{x}$ and δ by $\delta/2$ in (2)), for all $k \geq n_1$

$$\|x^{k+1} - \bar{x}\|^2 \leq \|x^k - \bar{x}\|^2 + \lambda_k^2 - \lambda_k \delta.$$

Adding the above inequality from $k = n_1$ to a sufficiently large n leads to a contradiction. Therefore, $\limsup_{k\to\infty} c^T x^k = v^*$. □

Finally, we prove the convergence of the method for an unbounded feasible region.

THEOREM 3. *If the feasible region is unbounded and $S^* \neq \emptyset$, then the sequence $\{x^k : k = 1, 2, \dots\}$ generated by the method converges to an optimal solution of (SIP).*

Proof. First, we prove that $\{x^k : k = 1, 2, \dots\}$ is bounded. Let $x^* \in S^*$ and K and θ be defined as in Lemma 1. Let $\lambda_0 = \sup\{\lambda_k : k = 1, 2, \dots\}$. As in the proof of Theorem 1, for all $k \geq K$

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 + 2\lambda_k^2 + 2\lambda_k d(x^k). \tag{11}$$

Adding (11) from $k = K$ to $k = K + l - 1$ for any $l \geq 1$,

$$\|x^{K+l} - x^*\|^2 \leq \|x^K - x^*\|^2 + 2\sum_{m=0}^{l-1} \lambda_{K+m}^2 + 2\sum_{m=0}^{l-1} \lambda_{K+m} d(x^{K+m}). \tag{12}$$

By Lemma 2 (a),

$$\begin{aligned}
&\sum_{m=0}^{l-1} \lambda_{K+m} d(x^{K+m}) \\
&\quad \leq \sum_{m=0}^{l-1} \lambda_{K+m}(\theta^m d(x^K) + (1-\theta)^{-2}\lambda_{K+m-1}) \\
&\quad \leq d(x^K)\lambda_0 \sum_{m=0}^{l-1} \theta^m + (1-\theta)^{-2} \sum_{m=0}^{l-1} \lambda_{K+m}\lambda_{K+m-1} \\
&\quad \leq d(x^K)\lambda_0(1-\theta)^{-1} + (1-\theta)^{-2} \sum_{m=0}^{l-1} (\lambda_{K+m}^2 + \lambda_{K+m-1}^2)/2.
\end{aligned} \tag{13}$$

It follows from (12), (13), and (B2) that $\{x^k : k = 1, 2, \dots\}$ is bounded. From Theorem 2 and the boundedness of $\{x^k : k = 1, 2, \dots\}$, there exists a subsequence $\{x^{k_j} : j = 1, 2, \dots\}$ that converges to an optimal solution of (SIP). The proof for the convergence of the entire sequence is same as that of Theorem 1, as we now know that $\{x^k : k = 1, 2, \dots\}$ is bounded and $\{x^{k_j} : j = 1, 2, \dots\}$ is convergent. □

We have proved that the sequence $\{x^k : k = 1, 2, \dots\}$ converges to an optimal solution of (SIP) provided an optimal solution exists. In the case that v^* is infinite or v^* cannot be attained, the sequence $\{x^k : k = 1, 2, \dots\}$ is unbounded,

$\limsup_{k\to\infty} c^T x^k = v^*$, and $\lim_{k\to\infty} d(x^k) = 0$. Thus the method solves (SIP) completely.

4. Computational Results

We use the method to solve the following semi-infinite linear program:

$$\begin{aligned} &\text{maximize } e^T x \\ &\text{subject to } (u_1^2, \cdots, u_n^2)x \le u^T A u \quad \text{for all } u \in S^{n-1} \\ &\qquad\qquad x_i \ge 0 \quad \text{for all } i = 1, \ldots, n, \end{aligned}$$

where $x \in R^n$, $e^T = (1, \ldots, 1)$, A is a given symmetric positive definite matrix, and $S^{n-1} = \{u \in R^n : \|u\| = 1\}$ is the unit sphere in R^n. It is known that a solution to this semi-infinite linear program is a solution to the educational testing problem (Fletcher, 1981). Letting $\bar{x} = (1, \ldots, 1)/\sqrt{n}$, one can verify that a constant τ satisfying Condition A is $\sqrt{n}$. Theoretically, the method is convergent as long as $\sum_{k=1}^{\infty} \lambda_k$ satisfies Condition B. Practically, the performance of the method depends on the choice of $\sum_{k=1}^{\infty} \lambda_k$. A class of infinite series satisfying Condition B is $\sum_{k=1}^{\infty} (\ln k)^{\gamma}/k$, where $\gamma \ge 0$. For a given problem, choosing a sufficiently large γ can accelerate convergence and prevent the method from stalling (see Table I). At the k-th iteration, the nonlinear subproblem to be solved in Step 1 is

$$\begin{aligned} &\max\{\max\{u^T(D(x^k + \lambda_k e) - A)u : u \in S^{n-1}\}, \\ &\quad \max\{-(x_i^k + \lambda_k) : i = 1, \ldots, n\}\}, \end{aligned}$$

where $D(x^k + \lambda_k e)$ is a diagonal matrix with diagonal entries $x_i^k + \lambda_k$, $i = 1, \ldots, n$. It is well known that solving $\max\{u^T(D(x^k+\lambda_k e) - A)u : u \in S^{n-1}\}$ is equivalent to finding the biggest eigenvalue and a unit eigenvector of $D(x^k + \lambda_k e) - A$. The subroutine RS (Smith *et al.*, 1970) is used to solve $\max\{u^T(D(x^k + \lambda_k e) - A)u : u \in S^{n-1}\}$ and it is assumed that RS can return "exact" eigenvalues and eigenvectors. Hence, the parameter α is set to zero and Step 2 is skipped. For simplicity, the starting point is $x_i = A(i,i)$, $i = 1, \ldots, n$. The stopping rule is $\|x^{k+1} - x^k\| < 10^{-8}$ or $k = 2000$. The method was coded in FORTRAN and the program was executed on a SUN Sparcstation ELC. Given

$$A = \begin{pmatrix} 9 & 0 & -1 & 2 & 4 \\ 0 & 10 & 1 & 3 & -1 \\ -1 & 1 & 15 & -5 & 0 \\ 2 & 3 & -5 & 16 & -2 \\ 4 & -1 & 0 & -2 & 12 \end{pmatrix}.$$

We have tested the method for $\lambda_k = \ln k/k$, and $\lambda_k = (\ln k)^2/k$. In addition, we have tested a mixed λ_k, i.e., $\lambda_k = (\ln k)^2/k$ if $k \le 300$ and $\lambda_k = 1/k$ if $k > 300$. Here the idea is to accelerate convergence by quickly reaching a neighborhood

TABLE I.

	$\lambda_k = \ln k/k$	$\lambda_k = (\ln k)^2/k$	mixed λ_k
final solution x_1	3.996552	3.999998	3.999951
final solution x_2	6.995968	6.999998	6.999943
final solution x_3	9.991046	9.999995	9.999873
final solution x_4	4.019066	4.000010	4.000271
final solution x_5	6.997337	6.999999	6.999962
final obj. value	31.999970	32.000000	32.000000
total iteration	2000	608	398
elapsed CPU (sec.)	7.50	2.29	1.50

of an optimal solution and then reducing step lengths. The results are shown in Table I.

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